Stiff stability of the hydrogen atom in dissipative Fokker electrodynamics

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(Received 11 January 2005; published 24 May 2005)

We introduce an *ad hoc* electrodynamics with advanced and retarded Liénard-Wiechert interactions plus the dissipative Lorentz-Dirac self-interaction force. We study the covariant dynamical system of the electromagnetic two-body problem, i.e., the hydrogen atom. We perform the linear stability analysis of circular orbits for oscillations perpendicular to the orbital plane. In particular, we study the normal modes of the linearized dynamics that have an arbitrarily large imaginary eigenvalue. These large eigenvalues are fast frequencies that introduce a fast (stiff) time scale into the dynamics. As an application, we study the phenomenon of resonant dissipation, i.e., a motion where both particles recoil together in a drifting circular orbit (a bound state), while the atom dissipates center-of-mass energy only. This balancing of the stiff dynamics is established by the existence of a quartic resonant constant that locks the dynamics to the neighborhood of the recoiling circular orbit. The resonance condition quantizes the angular momenta in reasonable agreement with the Bohr atom. The principal result is that the emission lines of quantum electrodynamics agree with the prediction of our resonance condition within 1% average deviation.

I. INTRODUCTION

In this paper we experiment with the stability analysis of the circular orbits of the electromagnetic two-body problem. The main motivation is to understand the complex dynamics described by the electromagnetic equations of motion that involve delay and third derivatives. We give a method to derive the linearized equations of motion in the neighborhood of the circular orbits of this implicitly defined dynamical system with delay. We introduce an *ad hoc* electromagneticlike setting that uses advanced and retarded Liénard-Wiechert interactions plus the dissipative Lorentz-Dirac self-interaction force [1], henceforth called the dissipative Fokker setting (DFS). We study in detail a specific feature of the tangent dynamics of the circular orbits of the two-body problem: the stiff normal modes of the linearized dynamics, which have an arbitrarily large imaginary eigenvalue. Last, we discuss an application for the hydrogen atom and the surprising predictions of stability analysis within the DFS; we predict several features of the Bohr atom [2] with high precision and qualitative detail. A subset of the emission lines predicted by the DFS agrees with the lines of quantum electrodynamics (QED) within 1% average deviation. There is also a surprising body of qualitative agreement with QED; (i) the emitted frequency is different from the orbital frequency; (ii) the stable orbits of the DFS have angular momenta that are multiples of a basic angular momentum. This basic angular momentum of the DFS agrees well with Planck's constant and depends only logarithmically on the mass of the heavier particle.

Dirac's 1938 fundamental work [1] on the electrodynamics of point charges gave complex and stiff delay equations that were seldom studied. Among the few models studied within Dirac's theory, the system of Eliezer's theorem [3–6] revealed a surprising dynamics; an electron moving in a Coulomb field with inclusion of self-interaction can never fall into the center of force by radiating energy. The result was generalized to motions in arbitrary attractive potentials [4], as well as to tridimensional motions with self-interaction in a Coulomb field [5,6], finding that only scattering states are possible. Since our model has Eliezer's problem as the infinite-mass limit, a finite mass for the proton is essential for a physically meaningful dynamics; if the proton has a finite mass, there is no inertial frame where it rests at all times, and this in turn causes delay because of the finite speed of light. It is widely known that QED gives a satisfactory and precise description of atomic physics, but the same is not popularly thought about atomic models based on classical electrodynamics. Since dynamical studies are still missing, clearly this complex dynamics needs to be investigated beyond our preliminary findings. Even though we are not trying to replace QED, our understanding of this two-body dynamics might prove useful for atomic physics, and perhaps we can understand QED as the effective theory of this complex stiff dynamics with delay. We shall describe the two-body motion in terms of the familiar *center-of-mass coordinates* and *coordinates of relative separation*, defined as the familiar coordinate transformation that maps the two-body Kepler problem into the one-body problem with a reduced mass. We stress that in the present relativistic motion the Cartesian center-of-mass vector is not ignorable, and it represents three extra coupled degrees of freedom. We introduce the concept of resonant dissipation to exploit this coupling and the many solutions that a delay equation can have. Resonant dissipation is the condition that both particles decelerate together, i.e., the center-of-mass vector decelerates, while the coordinates of relative separation perform an almost-circular orbit, despite of the energy losses of the metastable center-of-mass dynamics.

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Historically Nördstrom [7,8] suggested the use of advanced and retarded potentials in atomic physics already in 1920, but the self-interaction theory was problematic in 1920 and the idea disappeared. The theory of nonlinear dynamics was not out yet in 1938 when Dirac's theory for the electrodynamics of point charges appeared [1], neither in the glorious days of the twentieth century physics [9], such that the our present experiment is an application of modern nonlinear dynamics. Advanced interactions appeared again in 1945, when Wheeler and Feynman [10,11] gave an electrodynamics based on the postulate that every field is produced by charges located somewhere [12]. The theory was called action-at-a-distance electrodynamics [10,11], a theory where the isolated two-body problem is defined by Fokker's action

$$S_F = -\int m_1 ds_1 - \int m_2 ds_2 - e_1 e_2$$

$$\times \int \int \delta(\|\mathbf{x}_1 - \mathbf{x}_2\|^2) \dot{\mathbf{x}}_1 \cdot \dot{\mathbf{x}}_2 ds_1 ds_2, \qquad (1)$$

with $\mathbf{x}_i$, s_i, m_i, and e_i representing the four-position, the proper time, the mass, and the charge of particles $i=1,2$ respectively. In Eq. (1) the dot indicates the Minkowski scalar product of four vectors and double bars stand for the four-vector modulus [10,11]. Due to the similarities with the equations of motion of the DFS, the dynamical studies of the action-at-distance theory are relevant for the present work. For example, in the collision of two electrons with equations of motion determined by Eq. (1), the solution is determined by initial position and velocity only, as proved in Ref. [13] (a Banach-to-Banach contraction mapping proof for nonrunaway orbits). This suggests that we are dealing with a perfectly causal and well-posed dynamical system dressed in unusual form [14]. Driver's result [13] suggests that a dynamics with advance and delay is well-posed in the same way. The DFS presents exactly the same neutral-delay mathematical problem of any electromagneticlike model, as for example the problem with retarded-only fields of Refs. [15,16]. Fokker's action of Eq. (1) is used here to derive the sector of the DFS equations of motion determined by the semisum of Liénard-Wiechert fields. Last, advanced interactions appeared again in another work of Eliezer; a generalization of Dirac's covariant subtraction of electromagnetic infinities [17]. The resulting generalized electromagnetic settings include the advanced interactions naturally, and provide a testbed for future studies in electrodynamics [17]. Here, we shall keep to the DFS as a generic electromagneticlike example.

The road map for this paper is as follows: In Sec. IV we give the main technical part of the paper; we outline an economical method to derive the tangent dynamics of the circular orbit based on a quadratic expansion of the implicit light-cone condition. In this section we also take the stiff limit of the linear modes of the tangent dynamics. In Sec. V we give an application to atomic physics, by discussing a necessary condition for the state of resonant dissipation; This condition is heuristically expressed by a simple resonance condition that predicts the correct atomic scales. The earlier sections are a prelude to Sec. V. Section II is a review of the circular
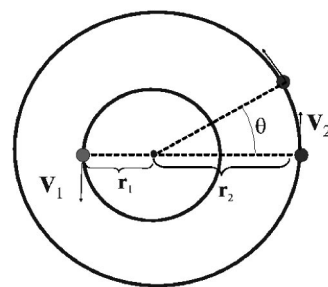

FIG. 1. The unperturbed circular orbit with the particles in diametral opposition at the same time in the inertial frame. Indicated is also the advanced position of particle 2 and the angle traveled during the light-cone time. The drawing is not to scale; the circular orbit of the proton has an exaggerated radius for illustrative purposes. Arbitrary units.

orbit solution, to be used in Sec. IV and in Sec. V. In Sec. III we build familiarity with Fokker's action of the action-at-a-distance electrodynamics as a prelude to the quadratic expansions needed for the linear stability analysis of Section IV. Section VI contains conclusions and a discussion. In Appendix A we discuss how the DFS can be fit into Dirac's electrodynamics of point charges. Last, in Appendix B we discuss the soft normal modes of the tangent dynamics.

II. THE CIRCULAR ORBIT SOLUTION

In this section we review the circular-orbit solution of the isolated electromagnetic two-body problem of the action-at-a-distance electrodynamics [18,19], to be used as the unperturbed orbit. For the isolated electromagnetic two-body problem, the tangent dynamics studied in the next section is straight Lyapunov stability analysis. In the DFS there is also a very small force along the orbital plane of the circular orbit, such that a nondrifting circular orbit is not a solution of the equations of motion. In the DFS the tangent dynamics is the starting point of a perturbation scheme to impose that the resulting dynamics is a drifting circular orbit (the state of resonant dissipation).

We use the index $i=1$ for the electron and $i=2$ for the proton, with masses m_1 and m_2, respectively, as in Eq. (1). We henceforth use units where the speed of light is $c=1$ and $e_1 = -e_2 \equiv -1$ (the electronic charge). The circular orbit is illustrated in Fig. 1, a motion of the two particles in concentric circles with the same constant angular speed and along a diameter. This dynamics satisfies the time-symmetric problem of Fokker's action (1) because the symmetric contributions from future and past generate a resulting force normal to the velocity of each particle [18,19]. The details of this relativistic orbit will be given now. The constant angular velocity is indicated by Ω, the distance between the particles in light cone is r_b, and $\theta \equiv \Omega r_b$ is the angle that one particle turns while the light emanating from the other particle reaches it (the light-cone time lag). The angle θ is the natural independent parameter of this relativistic problem. Each particle travels a circular orbit with radius and scalar velocity defined by

$$r_1 \equiv b_1 r_b,$$

$$r_2 \equiv b_2 r_b, \quad (2)$$

and

$$v_1 = \Omega r_1 = \theta b_1,$$

$$v_2 = \Omega r_2 = \theta b_2, \quad (3)$$

for the electron and for the proton, respectively. The condition that the other particle turns an angle θ during the light-cone time lag is [18]

$$b_1^2 + b_2^2 + 2b_1 b_2 \cos(\theta) = 1, \quad (4)$$

and is henceforth called the unperturbed light-cone condition. In Appendix B we calculate b_1 and b_2 in a power series of θ up to the fourth order. Last, because of the rotational invariance of Fokker's action, there is a conserved angular momentum perpendicular to the plane of the orbit that is evaluated in Ref. [18] to be

$$l_z = \frac{1 + v_1 v_2 \cos(\theta)}{\theta + v_1 v_2 \sin(\theta)}, \quad (5)$$

where the units of l_z are e^2/c; we are just using a unit system where $e^2 = c = 1$ [18]. Equation (4) restricts b_1 and b_2 to be less than 1 such that for small values of θ the angular momentum of Eq. (5) is of the order of $l_z \sim \theta^{-1}$. For orbits in the atomic magnitude, $l_z \simeq \theta^{-1}$ is about 1 over the fine-structure constant, $\alpha^{-1} = 137.036$. It is curious to notice that each (advanced and/or retarded) interaction term of Fokker's action, Eq. (9), evaluates exactly to $\frac{1}{2}\Omega l_z$ along a circular orbit, with l_z given by Eq. (5). This combination of angular momentum times the orbital frequency is reminiscent of the formal maneuvers of quantum mechanics.

III. FOKKER'S ACTION

We use Fokker's action in this work as a means to derive the sector of the DFS equations of motion determined by the semisum of the Liénard-Wiechert potentials. In the following we discuss the Lagrangian formalism of Fokker's action (1) as an introduction to our economical method to obtain the tangent dynamics by expanding this action to quadratic order. The delta function of Fokker's action (1) contains the retarded and the advanced light-cone contributions, and it is convenient to separate those two parts by factoring the argument of the delta function as

$$(t_1 - t_2)^2 - r_{12}^2 = [t_1 - t_2 - r_{12}][t_1 - t_2 + r_{12}], \quad (6)$$

where r_{12} stands for the Cartesian distance between particle 1 at time t_1 and particle 2 at time t_2, and each factor on the right-hand side of Eq. (6) is related to the advanced and the retarded light cones of particle 1, respectively. The delta function of a product argument is a sum of two delta functions, each multiplied by the respective Jacobian, such that the interaction term of Fokker's action (1) can be written as

$$VA = \int \frac{1}{2r_{12}} \delta(t_1 - t_2 - r_{12})(1 - \mathbf{v}_1 \cdot \mathbf{v}_2) dt_1 dt_2$$
$$+ \int \frac{1}{2r_{12}} \delta(t_1 - t_2 + r_{12})(1 - \mathbf{v}_1 \cdot \mathbf{v}_2) dt_1 dt_2, \quad (7)$$

where $\mathbf{v}_1$ henceforth stands for the Cartesian velocity of particle 1 at time t_1 and $\mathbf{v}_2$ henceforth stands for the Cartesian velocity of particle 2 at time t_2. We henceforth use the dot to indicate the scalar product of two Cartesian vectors, as already used in Eq. (7). Integration of each term of Eq. (7) over t_2 brings out another Jacobian factor and yields

$$VA = \int \frac{1}{2} \frac{(1 - \mathbf{v}_1 \cdot \mathbf{v}_{2a})}{r_{12}(1 + \mathbf{n}_{12a} \cdot \mathbf{v}_{2a})} dt_1 + \int \frac{1}{2} \frac{(1 - \mathbf{v}_1 \cdot \mathbf{v}_{2b})}{r_{12}(1 - \mathbf{n}_{12b} \cdot \mathbf{v}_{2b})} dt_1, \quad (8)$$

where $\mathbf{n}_{12a}$ and $\mathbf{n}_{12b}$ are unit vectors along the direction connecting the position of particle 1 at time t_1 to the position of particle 2 at time t_2 at the advanced and retarded times respectively, and $\mathbf{v}_{2a}$ and $\mathbf{v}_{2b}$ stand for the velocity of particle 2 at the advanced and retarded time t_2 respectively. Equation (8) is the most useful form of Fokker's interaction for our purposes. Notice that each term of Eq. (8) can be cast in the form

$$\frac{1}{2} \int \frac{(1 - \mathbf{v}_1 \cdot \mathbf{v}_{2c})}{r_{12}\left(1 + \dfrac{\mathbf{n}_{12} \cdot \mathbf{v}_{2c}}{c}\right)} dt_1 \equiv \frac{-1}{2} \int (V - \mathbf{v}_1 \cdot \mathbf{A}) dt, \quad (9)$$

where V and $\mathbf{A}$ are the Liénard-Wiechert scalar potential and the Liénard-Wiechert vector potential, respectively. We have introduced the quantity $c = \pm 1$ in the denominator of Eq. (9) such that $c = 1$ represents the advanced interaction while $c = -1$ represents the retarded interaction. The quantities of particle 2 in Eq. (9) are to be evaluated at a time t_2 defined implicitly by

$$t_2 = t_1 + \frac{r_{12}}{c}, \quad (10)$$

where $c = \pm 1$ describes the advanced and retarded light cones, respectively. Because of this decomposition of Fokker's interaction into V and $\mathbf{A}$ parts, we henceforth call Eq. (9) the VA interaction. A derivation of the Liénard-Wiechert potentials from Fokker's action and details such as the Darwin approximation are found in Ref. [20].

The stiff limit is determined by the largest-order derivative appearing in the linearized equations of motion of Appendix A. In this approximation, the contribution of the self-interaction force to the linearized dynamics about a circular orbit is simply given by the Abraham-Lorentz -Dirac force

$$\mathbf{F}_{rad} = \frac{2}{3}\dot{\mathbf{a}}. \quad (11)$$

The contribution of the other smaller terms will be given elsewhere.

IV. LINEAR STABILITY ANALYSIS

In this section we study the linear stability analysis of the circular orbits for displacements perpendicular to the orbital plane, henceforth called the z direction. We give an economical method to obtain these equations of tangent dynamics by expanding the implicit light-cone condition up to quadratic order. We start from the equations of motion of the isolated system, which are derived from Fokker's action (1) and yield the Liénard-Wiechert fields in the half-retarded plus half-advanced combination. This linearized z dynamics is uncoupled from the planar dynamics, and the linearized equations can de derived without the use of a symbolic software, as we explain in the following. The Cartesian coordinates of a transversely perturbed circular orbit are defined by

$$x_k + iy_k \equiv r_b d_k \exp(i\Omega t),$$

$$x_k - iy_k \equiv r_b d_k^* \exp(-i\Omega t), \qquad (12)$$

$$z_k \equiv r_b CSZ_k,$$

where $k=1$ for the electron and $k=2$ for the proton, Z_k is the small transverse perturbation, $d_1 \equiv b_1$ and $d_2 \equiv -b_2$ are defined from the two real parameters of Eq. (2), and Ω is the orbital frequency defined above Eq. (2). Last, in Eq. (12) C and S are defined by

$$C \equiv 1 + b_1 b_2 \theta^2 \cos(\theta), \qquad (13)$$

and

$$S \equiv 1 + b_1 b_2 \theta \sin(\theta). \qquad (14)$$

We henceforth introduce a scaled time $\tau \equiv \Omega t$. The linear stability analysis involves expanding the equations of motion to linear order in Z_k, which in turn is determined by the quadratic expansion of Fokker's action in Z_k. The main tool for expanding this quadratic action is the perturbed light-cone condition, Eq. (10), about the circular orbit (where $r_{12} = r_b$ is the constant circular lag). We introduce a function φ of the Z_1 and Z_2 perturbations by expanding the light-cone time t_2 as

$$t_2 \equiv t_1 + \frac{r_b}{c} + \frac{\varphi}{\Omega}. \qquad (15)$$

In the following we calculate this homogeneous functional φ of Z_1 and Z_2 up to quadratic order. The distance r_{12} entering Eq. (10) is to be evaluated from the position of particle 1 at time t_1, to the position of particle 2 at the time t_2 defined implicitly by Eq. (15). The coordinates of particle 2 at the time t_2 are defined implicitly by

$$x_2 + iy_2 = r_b d_2 \exp(i\tau_1 + ic\theta + i\varphi),$$

$$x_2 - iy_2 = r_b d_2^* \exp(-i\tau_1 - ic\theta - i\varphi), \qquad (16)$$

$$z_2 = r_b CSZ_2(\tau_1 + c\theta + \varphi) \equiv r_b CSZ_{2c},$$

where $c=1$ for the advanced time t_2 and $c=-1$ for the retarded time t_2. Notice that Eq. (16) defines the coordinates implicitly, because φ is a function of the deviations Z_1 and Z_2. Even though θ is small in applications of atomic physics, we stress that one should never expand in powers of θ; the correct infinitesimal quantity of the linear stability analysis is the size of the deviations from circularity and their homogeneous functions such as φ (expanding in θ produces the Darwin approximation [20]). This nonanalyticity will become clear after we show that the logarithm of θ appears. We therefore expand the advanced and/or retarded position Z_{2c} of particle 2 at the scaled time $\tau_1 + c\theta + \varphi$ in a Taylor series in φ about the advanced and/or retarded position $\tau_1 + c\theta$. It turns out that only the zeroth-order term appears in the action up to quadratic order. Because of this, the linearized equations involve only a constant shift, a considerable simplification. Substituting t_2 of Eq. (15) together with the position (16) of particle 2 into Eq. (10), and using the Pythagoras theorem for the distance r_{12} from particle 1 at time t_1 to particle 2 at time t_2 yields

$$r_{12}^2 = \left(r_b + r_b \frac{\varphi c}{\Omega r_b}\right)^2 = r_b^2 + r_b^2 C^2 S^2 (Z_1 - Z_{2c})^2. \qquad (17)$$

Notice that the Z variations decouple from the planar variations because there is no mixed linear term of Z times a linear perturbation of the planar coordinate in Eq. (17); these are naturally separated by the Pythagoras theorem. The planar perturbations enter in Eq. (17) as an added quadratic form, as given in the next section. It is convenient to define another function Φ by $\varphi \equiv \theta c CS\Phi$, such that Eq. (17) is a quadratic equation of Φ and the regular solution up to second order in Z_1 and Z_{2c} is

$$\Phi = \frac{CS}{2}(Z_1 - Z_{2c})^2. \qquad (18)$$

The coordinate Z_2 appears evaluated at the advanced and/or retarded time in Eq. (18), and to obtain the action up to quadratic terms it is sufficient to keep the first term $Z_{2c} = Z_2(\tau_1 + c\theta + \varphi) \simeq Z_2(\tau_1 + c\theta)$. Using the z-perturbed orbit defined by Eq. (16) to calculate the numerator of the VA interaction of Eq. (9) yields

$$(1 - \mathbf{v}_1 \cdot \mathbf{v}_{2c}) = 1 + \theta^2 \cos(\theta) b_1 b_2 - \theta^2 C^2 S^2 \dot{Z}_1 \dot{Z}_{2c}, \qquad (19)$$

and the denominator of the VA interaction of Eq. (9) is

$$r_{12}(1 + \mathbf{n}_{12c} \cdot \mathbf{v}_{2c}/c) = r_b S[1 + C\Phi + \theta c C^2 S(Z_1 - Z_{2c})\dot{Z}_{2c}]. \qquad (20)$$

Notice that the quadratic term $Z_{2c}\dot{Z}_{2c}$ on the right-hand side of Eq. (20) can be dropped because it represents an exact gauge that does not affect the Euler-Lagrange equations of motion, such that

$$r_{12}(1 + \mathbf{n}_{12c} \cdot \mathbf{v}_{2c}/c) \approx r_b S[1 + C\Phi + \theta c C^2 S Z_1 \dot{Z}_{2c}], \qquad (21)$$

where the equivalence sign $\approx$ henceforth means equivalent up to a gauge term of second order. Even if a quadratic gauge term appears in the denominator, in an expansion up to quadratic order it would still produce a gauge and therefore it can be dropped directly from the denominator. One should be careful not to do this with linear gauges, which appear only in the planar stability analysis to be considered else-

where. In this way, the expansion up to second order of the VA interaction of Eq. (9) is simply

$$VA \approx \left(\frac{C}{2r_bS}\right)\left\{1 - \theta^2CS^2\dot{Z}_1\dot{Z}_{2c} - \frac{C^2S}{2}(Z_1 - Z_{2c})^2\right.$$

$$\left. - \theta cC^2SZ_1\dot{Z}_{2c}\right\}. \tag{22}$$

Last, we need the kinetic energy along the z-perturbed circular orbit, which we express in terms of Z_1 of definition (12) as

$$T_1 = -m_1\sqrt{1 - v_1^2} = -\frac{m_1}{\gamma_1}\sqrt{1 - \gamma_1^2C^2S^2\theta^2\dot{Z}_1^2}, \tag{23}$$

where the dot means derivative with respect to the scaled time τ, $\gamma_1^{-1} \equiv \sqrt{1 - v_1^2}$, and we have used $\Omega r_b = \theta$. The expansion of Eq. (23) up to second order is

$$T_1 = \left(\frac{C}{r_bS}\right)\left\{\frac{-r_bSm_1}{C\gamma_1} + \frac{\epsilon_1}{2}\dot{Z}_1^2 + \cdots\right\}, \tag{24}$$

where $\epsilon_1 \equiv m_1r_b\gamma_1\theta^2CS^3$ is calculated with Eq. (B1) to be

$$\epsilon_1 \equiv \frac{C}{b_1}\{[C^2 + \theta^2S(S-1)][b_1 + b_2\cos(\theta)]$$

$$+ S[\theta\sin(\theta) - \theta^2\cos(\theta)]b_2\}. \tag{25}$$

We are ready to derive the Euler-Lagrange equation of motion for particle 1 of the isolated two-body problem using the quadratic Lagrangian

$$L_1 = T_1 + VA_{c=1} + VA_{c=-1}. \tag{26}$$

This equation of motion is

$$\epsilon_1\ddot{Z}_1 = -\frac{C^2S}{2}(2Z_1 - Z_{2+} - Z_{2-}) - \frac{\theta C^2S}{2}(\dot{Z}_{2+} - \dot{Z}_{2-})$$

$$- \frac{\theta^2CS^2}{2}(\ddot{Z}_{2+} + \ddot{Z}_{2-}). \tag{27}$$

Notice that the term on the left-hand side of Eq. (27) can be written as

$$\epsilon_1\ddot{Z}_1 = r_b^2S^2m_1\gamma_1\Omega^2CS\ddot{Z}_1 = r_b^2S^2\frac{dp_z}{dt}, \tag{28}$$

which is proportional to the force along the z direction. According to the prescription of the DFS, we shall add the following self-interaction term to the right-hand side of Eq. (27):

$$r_b^2S^2\mathbf{F}_{rad} = \frac{2}{3}CS^3\dddot{Z}_1, \tag{29}$$

where the triple dot means three derivatives with respect to the scaled time and we have used Eq. (11). The full linearized equation of motion for Z_1 is

$$\epsilon_1\ddot{Z}_1 = \frac{2}{3}CS^3\dddot{Z}_1 - \frac{C^2S}{2}(2Z_1 - Z_{2+} - Z_{2-}) - \frac{\theta C^2S}{2}(\dot{Z}_{2+} - \dot{Z}_{2-})$$

$$- \frac{\theta^2CS^2}{2}(\ddot{Z}_{2+} + \ddot{Z}_{2-}). \tag{30}$$

The linearized equation for Z_2 is completely analogous and is obtained by interchanging Z_1 by Z_2 and ϵ_1 by ϵ_2 in Eq. (30). The general solution of a linear delay equation can be obtained by Laplace transform [21] and is a linear combination of the following normal mode solutions. A normal mode solution is obtained by substituting $Z_1 = A\exp(p\tau)$ and $Z_2 = B\exp(p\tau)$ into the two linearized equations, and requires the vanishing of the following 2×2 determinant:

$$\det Z \equiv \begin{vmatrix} C^2S + \epsilon_1p^2 - \frac{2}{3}CS^3\theta^3p^3 & G(\theta,p) \\ G(\theta,p) & C^2S + \epsilon_2p^2 - \frac{2}{3}CS^3\theta^3p^3 \end{vmatrix}, \tag{31}$$

where $G(\theta,p) \equiv (C^2S - CS^2\theta^2p^2)\cosh(p\theta) + C^2Sp\theta\sinh(p\theta)$. Two kinds of limits are interesting for the infinite-dimensional formal collection of normal modes of Eq. (31); (i) the four soft Coulomb modes obtained by expanding Eq. (31) in powers of θ for small values of p, as discussed in Appendix B, and (ii) the stiff limit obtained when $p\theta$ is large, such that the hyperbolic functions of the $G(\theta,p)$ acquire a large magnitude [22]. In the following we use the zeroth-order term of the expansion for b_1 and b_2 given in Appendix B to evaluate the determinant (31)

$$\epsilon_1 = \frac{M}{m_2} + O(\theta^2),$$

$$\epsilon_2 = \frac{M}{m_1} + O(\theta^2),$$

$$C = 1 + O(\theta^2),$$

$$S = 1 + O(\theta^2). \tag{32}$$

For small θ, the second-order and higher even-order terms of Eq. (32) give only a small correction. Substituting Eq. (32) into Eq. (31) and defining $p \equiv \lambda/\theta$, we obtain

$$\frac{\mu\theta^4}{M\lambda^4}(\det Z) = 1 - \frac{2}{3}\theta^2\lambda + \frac{4}{9}\frac{\mu}{M}\theta^4\lambda^2 - \frac{\mu\theta^4}{M}$$

$$\times\left[\left(1 - \frac{1}{\lambda^2}\right)\cosh(\lambda) - \frac{1}{\lambda}\sinh(\lambda)\right]^2, \tag{33}$$

where we have dropped small $O(\theta^2)$ terms. The stiff-mode condition defined by Eq. (31) (det $Z=0$) is

$$1 - \frac{2}{3}\theta^2\lambda + \frac{4\mu}{9M}\theta^4\lambda^2 - \frac{\mu\theta^4}{M}\left(1 - \frac{1}{\lambda^2} + \frac{1}{\lambda^4}\right)\cosh^2(\lambda)$$

$$+ \frac{1}{\lambda}\left(1 - \frac{1}{\lambda^2}\right)\sinh(2\lambda) = 0. \tag{34}$$

For future reference we give also the stiff limit for the z-tangent dynamics without the self-interaction terms, which

is obtained from Eq. (27), and the corresponding equation for particle 2

$$1 - \frac{\mu\theta^4}{M}\left(1 - \frac{1}{\lambda^2} + \frac{1}{\lambda^4}\right)\cosh^2(\lambda) + \frac{1}{\lambda}\left(1 - \frac{1}{\lambda^2}\right)\sinh(2\lambda) = 0. \tag{35}$$

V. THE STIFF STABILITY OF THE HYDROGEN ATOM

We are interested in finding motions where the particles recoil together while staying in the neighborhood of a drifting circular orbit, i.e., the state of resonant dissipation. The need for a resonance becomes obvious in the following perturbative scheme. (i) We take the circular orbit as the unperturbed state. (ii) We substitute the circular orbit plus a perturbation into the equations of motion of the DFS and take the linearized equations of motion. The circular orbit is not an exact solution of the DFS equations of motion, because of the small forcing coming from the third derivatives. This perturbative scheme yields linear delay equations with a small forcing term along the orbital plane. It is then possible to show by averaging [23] that a weakly accelerated drifting circular orbit is never a solution to these linear equations. Therefore, a bifurcation of the circular orbit must happen and a nonlinear term must be important to balance the small dissipative forcing, if the state of resonant dissipation is to be attained. In the following, we postulate that this resonance happens at a quartic order. By inspection, one finds that only resonance conditions involving the stiff modes can be satisfied in the atomic magnitude. In the following we study the consequences that along some special circular orbits such balancing mechanism is established by the existence of a quartic resonant constant of motion. To discuss this stability by resonance we need some results of the tangent dynamics along the orbital plane. This more elaborate tangent dynamics is derived in a way analogous to Sec. V and shall be given elsewhere; here, we give only the main results. The stiff limit for the equal-mass two-body problem with retarded and advanced fields is studied in Ref. [22], and in the following we give the generalization of these results for the case of arbitrary masses.

Up to linear order, the tangent dynamics along the orbital plane is decoupled from the z dynamics of Sec. V. To study this planar tangent dynamics, it is convenient to describe the orbit along the $z=0$ plane using gyroscopic coordinates

$$x_k + iy_k \equiv r_b \exp(i\Omega t)[d_k + \eta_k],$$

$$x_k - iy_k \equiv r_b \exp(-i\Omega t)[d_k + \xi_k], \tag{36}$$

where η_k and ξ_k are complex numbers defining the perturbation of the circularity and the d_k are defined below Eq. (12). Because x_k and y_k are real we should have $\eta_k = \xi_k^*$, but a convenient way to minimize the quadratic functional of Fokker's action is to treat η_k and ξ_k as independent functions. To fix ideas we start from the stability of the isolated two-body system, and again we define the normal-mode eigenvalue by $\lambda\Omega/\theta$, i.e., every coordinate perturbation oscillates in time as

$\exp(\lambda\Omega t/\theta)$ (λ is an arbitrary complex number). The limiting form of the planar characteristic equation for the isolated different-mass case is

$$\left(\frac{\mu\theta^4}{M}\right)\cosh^2(\lambda) = 1, \tag{37}$$

where μ is the reduced mass and $M \equiv m_1 + m_2$ [for the equal-mass case, our general Eq. (37) reduces to Eq. (15) of Ref. [22]]. Along circular orbits both the planar and the perpendicular linearized equations share the same limiting characteristic Eq. (37), as can be checked with Eq. (34). For hydrogen (μ/M) is a small factor of about (1/1824). It is important to understand the structure of the roots of Eq. (37) in the complex λ plane, specially for θ of the order of the fine structure constant. The very small parameter $\mu\theta^4/M \sim 10^{-13}$ multiplying the squared hyperbolic cosine on the left-hand side of Eq. (37) determines that $\sigma \equiv |\text{Re}(\lambda)| \simeq \ln(\sqrt{4M/\mu\theta^4})$. For the first 13 excited states of hydrogen this σ is in the interval $14.2 < |\sigma| < 18.2$. The imaginary part of λ can be an arbitrarily large multiple of π, such that the general solution to Eq. (37) is

$$\lambda = \pm(\sigma + i\pi q), \tag{38}$$

where q is an arbitrary integer. The plus or minus sign of Eq. (38) is related to the time reversibility of the isolated two-body system, a symmetry that is broken by radiation. This same exact phenomenon happens for the z direction. Next, we include the dissipation of the DFS, i.e., the Lorentz-Dirac self-interaction, a calculation performed by adding the self-interaction force to the equations of motion of the isolated system. Here, we give only the characteristic planar equation up to $O(1/\lambda^4)$

$$\left(1 + \frac{7}{\lambda^2} + \frac{5}{\lambda^4}\right)\left(\frac{\mu\theta^4}{M}\right)\cosh^2(\lambda) = 1 - \frac{2}{3}\theta^2\lambda + \frac{1}{9}\theta^4\lambda^2$$

$$+ \left(\frac{1}{\lambda} + \frac{5}{\lambda^3}\right)\left(\frac{\mu\theta^4}{M}\right)\sinh(2\lambda) + \cdots. \tag{39}$$

It is remarkable that Eqs. (39) and (34) differ only at the terms of $O(1/\lambda)$ and at the terms of type $\theta^4\lambda^2$, which describe small corrections for σ in the atomic range. The linear term on the right-hand side of Eqs. (34) and (39) with the 2/3 coefficient is due to the self-interaction force. This dissipative term breaks the time-reversal symmetry of Eq. (37), and the roots of Eqs. (34) and (39) no longer come in plus or minus pairs. Let λ_{xy} be a root of Eq. (39) with positive real part and λ_z be a root of Eq. (34) with a negative real part. In the stiff limit these are both near one of the limiting roots (38) and can be expressed as

$$\lambda_{xy} \equiv (\sigma + \pi q i + i\epsilon_1),$$

$$\lambda_z \equiv -(\sigma + \pi q i + i\epsilon_2), \tag{40}$$

where the small perturbations ϵ_1 and ϵ_2 are so far two arbitrary complex numbers. The second-order balancing process studied here involves the interaction of a z mode with a planar mode, in the same way used in Refs. [15,16]. This is because if the atom is to recoil like a rigid body, one expects

the stiff dynamics to perform fast-spinning toroidal motions about the slow circular orbit.

We henceforth assume heuristically that the state of resonant dissipation is formed in a bifurcation involving perturbations along two special linear modes of the tangent dynamics. We take a perpendicular normal mode of Eq. (34) and a planar normal mode of Eq. (39), with eigenvalues λ_z and λ_{xy}, respectively. The coordinate of the planar normal mode is a linear combination of the four $\eta\xi$ gyroscopic coordinates: $u \equiv a_{1k}\eta_k + b_{1k}\xi_k$, while the coordinate of the perpendicular z normal mode is $Z \equiv b_1 z_1 + b_2 z_2$. Using the normal-mode conditions $\theta\dot{u} = \Omega\lambda_{xy}u$ and $\theta\dot{Z} = \Omega\lambda_z Z$, one can show that the quadratic form uZ is a complex amplitude that oscillates harmonically with the beat frequency $(\lambda_{xy} + \lambda_z)\Omega/\theta = i(\epsilon_1 - \epsilon_2)\Omega/\theta$. Our resonance condition is to choose these two eigenvalues such that

$$\mathrm{Re}(\lambda_{xy} + \lambda_z) = 0. \tag{41}$$

Condition (41) avoids that the modulus of the amplitude uZ has an exponential growth. We shall see that condition (41) is satisfied only for special discrete values of θ. Since condition (41) must be satisfied, we henceforth assume that ϵ_1 and ϵ_2 are real numbers, as any excess real part in Eq. (40) can be absorbed in the definition of σ. Condition (41) is also the necessary condition to construct a resonant constant in the neighborhood of the circular orbit; because Fokker's action is real, λ_z^* and λ_{xy}^* are also eigenvalues to Eqs. (34) and (39), respectively, with complex conjugate normal-mode coordinates. Condition (41) then implies the usual necessary condition for a resonant constant

$$\lambda_{xy} + \lambda_z + \lambda_{xy}^* + \lambda_z^* = 0, \tag{42}$$

as discussed in Refs. [16,24]. Using these complex conjugate normal-mode coordinates and Eq. (41), one can show that the following quartic form is a constant of the motion up to higher order terms [16,24]:

$$C \equiv |u|^2 |Z|^2 + \cdots. \tag{43}$$

The quartic function of Eq. (43) is constant because it is the squared modulus of the harmonic amplitude $uZ = \sqrt{C}\exp[i(\epsilon_1 - \epsilon_2)\Omega t/\theta]$. This necessary condition and the continuation of the leading term (43) to an asymptotic series is discussed in Ref. [16].

The root-searching problem of Eq. (41) is well posed and for each integer q conditions (34) and (39) together with Eq. (40) determine a unique θ as a function of q, i.e., θ is quantized by the integer q that appears naturally in Eq. (40). An asymptotic solution to condition (40) can be obtained by expanding Eqs. (34) and (39) up to quadratic order in ϵ_1 and ϵ_2 while treating σ as an approximate constant. This approximation determines the following discrete values for θ:

$$\theta^2 = \frac{6(\pi^2 q^2 - \sigma^2)}{\sigma(\pi^2 q^2 + \sigma^2)^2}, \tag{44}$$

and

TABLE I. Numerically calculated angular momenta $l_z = \theta^{-1}$ in units of e^2/c, the orbital frequencies in atomic units $(137\theta)^3$, the circular lines of QED in atomic units $w_{QED} \equiv (1/2)\{(1/k^2) - [1/(k+1)^2]\}$, the emission frequencies of the DFS in atomic units $w_{DF} \equiv (137\theta)^3 + 137^3\theta^2(\epsilon_1 - \epsilon_2)$, and the values of the integer q of Eq. (40).

$l_z = \theta^{-1}$	$(137\theta)^3$	w_{QED}	w_{DF}	q
161.94	6.054×10^{-1}	3.750×10^{-1}	3.655×10^{-1}	32
283.52	1.128×10^{-1}	6.944×10^{-2}	6.774×10^{-2}	55
398.06	4.077×10^{-2}	2.430×10^{-2}	2.462×10^{-2}	76
520.29	1.826×10^{-2}	1.125×10^{-2}	1.110×10^{-2}	98
638.53	9.876×10^{-3}	6.111×10^{-3}	6.038×10^{-3}	119
752.27	6.039×10^{-3}	3.685×10^{-3}	3.710×10^{-3}	139
872.68	3.868×10^{-3}	2.406×10^{-3}	2.387×10^{-3}	160
988.16	2.664×10^{-3}	1.640×10^{-3}	1.650×10^{-3}	180
1110.15	1.879×10^{-3}	1.173×10^{-3}	1.168×10^{-3}	201
1226.95	1.392×10^{-3}	8.678×10^{-4}	8.677×10^{-4}	221
1344.30	1.058×10^{-3}	6.600×10^{-4}	6.615×10^{-4}	241
1462.14	8.226×10^{-4}	5.136×10^{-4}	5.153×10^{-4}	261
1580.44	6.513×10^{-4}	4.076×10^{-4}	4.090×10^{-4}	281

$$(\epsilon_1 - \epsilon_2) = \frac{4\pi q(3\sigma^2 - \pi^2 q^2)}{\sigma(\sigma^2 + \pi^2 q^2)^2}. \tag{45}$$

According to QED, the circular Bohr orbits have maximal angular momenta for that quantum number, and a radiative selection rule ($\Delta l = \pm\hbar$) restricts the decay from level $k+1$ to level k only, i.e., circular orbits emit the first line of each spectroscopic series (Lyman, Balmer, Ritz-Paschen, Brackett, etc.), henceforth called the QED circular line. We have solved Eqs. (34), (39), and (40) with a Newton method in the complex λ plane. Every angular momentum $l_z = \theta^{-1}$ determined by Eq. (41) has a value in the correct atomic magnitude ($\theta^{-1} \gtrsim 137.0$); the first resonance appears at $q=5$ for $\theta^{-1} = 252.4$ and the minimum value $\theta^{-1} = 48.52$ is attained at $q=7$, then θ^{-1} increases monotonically with q. The subset of Table I has frequencies w_{DF} surprisingly close to the QED lines. These lines are for q approximately equal to an integer multiple of the integer part of 2σ. We conjecture here that among the resonances satisfying the necessary condition (41), only some have $|u|^2$ depending on the translation-invariant quantities $(\xi_1 - \xi_2)$ and $(\eta_1 - \eta_2)$ to allow a recoiling translation [23]. In our description the emission mechanism is at a frequency equal to the orbital frequency Ω corrected by the frequency of the complex amplitude uZ defined above Eq. (43), as we explain below. The numerically calculated angular momenta $l_z = \theta^{-1}$ for this select subset are given in Table I, along with the orbital frequency in atomic units $(137^3\Omega)/\mu = (137\theta)^3$, the QED first frequency of the series in atomic units $w_{QED} \equiv (1/2)\{1/k^2 - [1/(k+1)^2]\}$, and the frequency predicted by the dissipative Fokker model $w_{DF} \equiv (137\theta)^3 + 137^3\theta^2(\epsilon_1 - \epsilon_2)$. We list only the first 13 lines, which are the experimentally observable, but we tested the agreement of the numerical calculations of the Newton method with up to the 40th circular line predicted by QED.

Beyond that, the asymptotic formula (44) shows that the agreement is essentially for any integer k because substitution of $q=[2\sigma]k$ into Eq. (44) yields

$$\theta^{-1} = \sqrt{\frac{2\pi^2}{3}}\sigma^{3/2}k \sim 137.9k, \qquad (46)$$

to be compared with the 137.036 of QED. The agreement for any integer k suggests that Eqs. (34) and (39) describe a linear problem that is equivalent to Schrödinger's equation (linear operators with the same spectrum are equivalent).

In the DFS the interaction with a distant particle involves half the retarded Liénard-Wiechert potential plus half the advanced Liénard-Wiechert potential (henceforth called the semisum). This semisum yields a radiation magnetic field for the electron of (the far-magnetic field)

$$\mathbf{B}_{rad} = \frac{(\mathbf{a}_- \times \hat{\mathbf{n}}_-)}{2(1-\hat{\mathbf{n}}_- \cdot \mathbf{v}_-)^2 r} - \frac{(\mathbf{a}_+ \times \hat{\mathbf{n}}_+)}{2(1+\hat{\mathbf{n}}_+ \cdot \mathbf{v}_+)^2 r}, \qquad (47)$$

where $\mathbf{v}$ and $\mathbf{a}$ are the electronic velocity and acceleration, $\hat{\mathbf{n}}$ is a unit vector from the electron to the observation point, the subindex minus sign indicates evaluation on the retarded light cone, and the subindex plus sign indicates evaluation on the advanced light cone. These two light cones are defined by $t_{\pm}=t\pm(r-\hat{\mathbf{n}}_{\pm}\cdot\mathbf{y})$, where $\mathbf{y}$ stands for the electron's position. Along a precise circular orbit the first approximation to Eq. (47) has a zero spatial average. For the next term we avoid the Page expansion of Appendix A, because the deviating arguments are large; we approximate the size of Eq. (47) by expanding the denominators of Eq. (47), yielding the quadratic function

$$\mathbf{B}_{rad}^{(1)} \simeq \frac{2(\hat{\mathbf{n}} \cdot \mathbf{v})(\mathbf{a} \times \hat{\mathbf{n}})}{r}. \qquad (48)$$

We can estimate $\mathbf{B}_{rad}^{(1)}$ of Eq. (48) by noticing that along the $\hat{\mathbf{n}}_{\pm}=\hat{\mathbf{x}}$ direction of the unperturbed plane this quadratic functional contains a product of the z perturbed coordinate times the x perturbed coordinate, i.e., the u and Z perturbations explained above Eq. (43). Translating the u mode to Cartesian coordinates with Eq. (36), we obtain

$$\mathbf{B}_{rad}^{(1)} \propto \frac{2uZ}{r} \exp(i\Omega t). \qquad (49)$$

According to Eq. (49), the frequency of the emission line is equal to Ω plus the frequency of the uZ amplitude

$$w_{DF} = \Omega + (\epsilon_1 - \epsilon_2)\Omega/\theta, \qquad (50)$$

with Ω given by Eq. (B6). Notice that the emitted frequency of the DFS is naturally different from the orbital frequency. The fact that the emission frequency of hydrogen is different from the orbital frequency is a famous conundrum. The emission frequency of Eq. (50) contains differences of eigenvalues of the linear operator of Eqs. (39) and (34) and is strikingly similar to the Rydberg-Ritz combinatorial principle of quantum mechanics for the emission lines.

VI. CONCLUSIONS AND DISCUSSION

In the limit where the proton has an infinite mass, the concept of resonant dissipation loses meaning because the center-of-mass coordinate no longer plays a dynamical role. In this singular limit, there is a Lorentz frame where the proton rests at the origin at all times, and the field at the electron reduces to a simple Coulomb field in the DFS. The two-body dynamics in the DFS reduces then to the dynamical system of Eliezer's theorem; self-interaction plus a Coulomb field acting on the electron [3,4]. We repeat this correct dynamics because it is very unpopular [3–6]; with inclusion of self-interaction, it is impossible for the electron to "spiral into the proton." Neither bound states nor dives are possible; only scattering states exist. This result is in surprising agreement with our formula (46) for the quantized angular momenta; if the mass of the proton is set infinite in Eq. (46), the quantized angular momenta become infinite logarithmically, θ goes to zero, and the particles are unbound at an infinite distance! One accomplishment of the present work is to recognize that only the two-body problem can produce a physically sensible electromagneticlike model. Even though there is a dependence on the mass in Eq. (46), the logarithm of the mass ratio times θ^4 makes the theory very insensitive to this mass ratio, such that the deuterium and the muonium have essentially the same quantized angular momenta, in reasonable agreement with QED. Qualitative disagreement would need an exponentially massive charged particle. Fortunately for our present theory, such particle does not exist in nature.

Another qualitative dynamical picture is suggested by Eliezer's result [3,4]; the dynamical phenomenon that the electron always turns away from the proton along unidimensional orbits suggests that colinear orbits are the natural attractors of the dissipative dynamics (a ground state with zero angular momentum!). Along such orbits, the heavy particle (the proton) moves in a non-Coulombian way and the self-interaction provides the repulsive mechanism that avoids the collision at the origin. This is again in agreement with the Schrödinger theory, where the ground state has a zero angular momentum. Again, the infinite-mass case produces unphysical dynamics; the electron turns away but then it runs away [4]. It remains to be researched if the two-body case has a physical orbit for zero-angular momentum orbits.

The theory of normal forms for delay equations is studied in Ref. [25]. An analogous mathematical phenomenon is the finite-dimensional center manifold for equations with advance and delay studied in connection with discrete shocks in the conservation laws of Refs. [26,27]. These conservation laws are similar to Dirac's relativistic Schrödinger's equation, and this would be a natural bridge to QED. Detailed construction of the resonant normal form is also needed to discuss the width of the emission lines. In the dynamical process of resonant exchange, the sharp line is emitted while the dynamics is locked to the neighborhood of the resonant orbit, which according to QED is a lifetime of about 10^6 turns in the hydrogen atom (10^{-9} s). We conjecture that when the metastable orbit breaks down, the dynamics falls into the next metastable attracting orbit; another circular orbit, or into the ground state [23].

The stiff modes of Eq. (40) introduce a fast (stiff) time scale with a frequency of the order of $\sigma/\theta \simeq 1400$ times the

orbital frequency, such that the time for a stiff jump of the dynamics is $1/1400$ times the orbital period, or 10^{-18} s! After this fast time scale the resonance essentially locks the dynamics to the neighborhood of the metastable resonant orbit. The fact that the equations of electrodynamics describe stiff jumps in the phase space is largely unexplored in the light of modern applied mathematics, mainly due to the complexity involved. The dynamics starting from an asymptotic resonant orbit to another of a neighboring q is certainly described by a stiff jump, as expected generically from any stiff equation. In Ref [28], the much simpler Van der Pol oscillator is worked out in detail as an example of an equation of Lienard type that exhibits stiff jumps. In quantum mechanics one seems to need the problematic concept of an "instantaneous quantum jump," to describe the stiff passage from one quantum state to another. It appears that classical electrodynamics prescribes exactly this qualitative phenomenon: a quasi-instantaneous fast dynamics.

The dynamics in the DFS solves several conundrums of the classical hydrogen atom and is similar to QED in many ways: (i) The radiated frequency is not equal to the orbital frequency (it is lesser than the orbital frequency; see Table I). (ii) The resonant orbits are naturally quantized by integers and the radiated frequencies agree with the Bohr circular lines within 1% average deviation. (iii) The ratio of the emitted frequency to the orbital frequency is in reasonable agreement with QED. (iv) The angular momenta of the resonant orbits are naturally quantized with the correct Planck's constant. (v) The stability analysis uses a linear dynamical system with delay, a dynamical system that needs an initial function as the initial condition, just like Schrödinger's equation. The emitted frequencies are then given by a difference of two eigenvalues of this linear operator, like the Rydberg-Ritz combinatorial principle of quantum physics. (vi) The eigenvalues of our linear operator have a large magnitude that does not appear in the frequency. This large magnitude is given by a logarithm, just like in the divergent perturbation theory for the Lamb shift of QED.

Recognizing the correct qualitative dynamics with the concept of resonant dissipation has taken us very far; the stability analysis indicated the need for resonances, and these turned out to be satisfied only for the stiff modes and precisely in the atomic magnitude! The stiff modes also provide a natural integer to label the resonant orbits. We selected the values of q among the larger set predicted by the necessary condition (41), showing that Eq. (41) is not in disagreement with QED. A sufficient condition should be part of the extra work to understand the unfolding of the bifurcation leading to the state of resonant dissipation. The large body of qualitative and quantitative agreement suggests that an extensive study of electromagneticlike models [17], of which the DFS is only a generic example, could offer an explanation of QED in terms of a stiff dynamical system with third derivatives and delay.

APPENDIX A: PHYSICAL JUSTIFICATION OF THE DFS

In Dirac's theory [1] the self-interaction is given by the sourceless combination of half of the retarded Liénard-Wiechert self-potential minus half of the advanced Liénard-Wiechert self-potential, i.e., the semidifference [1]. This gives the following concise description of the DFS: Charges interact with themselves via the semidifference of Liénard-Wiechert self-potentials and with other charges via the semisum of Liénard-Wiechert potentials. In the following we try to fit our *ad hoc* DFS into Dirac's theory as an effect of the physical boundaries on the fields. Dirac's electrodynamics of point charges [1] uses the retarded potential $F_{\mu k,ret}^{\nu}$ produced by each particle k and an incident free field $\tilde{F}_{\mu,in}^{\nu}$. In Dirac's theory the electron and the proton of a hydrogen atom have the following equations of motion [1]:

$$m_1\dot{v}_{1\mu} - \frac{2}{3}\ddot{v}_{1\mu} - \frac{2}{3}\|v_1\|^2 v_{1\mu} = -(F_{\mu,in}^{\nu} + F_{\mu2,ret}^{\nu})v_{1\nu},$$

$$(A1)$$

$$m_2\dot{v}_{2\mu} - \frac{2}{3}\ddot{v}_{2\mu} - \frac{2}{3}\|v_2\|^2 v_{2\mu} = (F_{\mu,in}^{\nu} + F_{\mu1,ret}^{\nu})v_{2\nu},$$

where double bars stand for the Minkowski scalar product, the electron and the proton have charges -1 and 1, respectively, and the speed of light is $c=1$. Since the DFS uses the semisum instead of the retarded-only potential, from the perspective of Dirac's theory this demands the following constraints on the free field $F_{\mu,in}^{\nu}$:

$$F_{\mu,in}^{\nu}(x_1(t)) = \frac{1}{2}[F_{\mu2,adv}^{\nu}(x_1(t)) - F_{\mu2,ret}^{\nu}(x_1(t))], \quad (A2)$$

$$F_{\mu,in}^{\nu}(x_2(t)) = \frac{1}{2}[F_{\mu1,adv}^{\nu}(x_2(t)) - F_{\mu1,ret}^{\nu}(x_2(t))], \quad (A3)$$

where the field of each particle is to be evaluated along the trajectory of the other particle, as indicated by the parenthesis after each field. Since both the advanced and the retarded fields satisfy Maxwell's equations, the semidifference is a free field, as assumed. The incident wave can be generated by the boundary conditions on the fields. For example, the reflections of the radiation by other atoms of a diluted gas could play the role of such a boundary condition.

The semidifference evaluated at the particle itself is the familiar self-interaction of the Dirac theory [1], and Eqs. (A2) and (A3) have instead the semidifference evaluated at the position of the other particle. Using the Page expansion of the Liénard-Wiechert fields, we find that the electric field of this semidifference is approximated by the third derivative of the other particle's coordinate, as discussed in Refs. [15,16]. In this approximation with the Page series [15,16], the incident electric field evaluated at the proton, Eq. (A3), is

$$E(x_2(t)) \simeq \frac{2}{3}\dddot{x}_1. \tag{A4}$$

Along the unperturbed orbit of Fig. 1, Eq. (A4) is an electromagnetic field rotating at the orbital frequency. For orbits in the atomic magnitude the electric field of Eq. (A4) has an intensity that turns out to be of the order of the polarized vacuum of QED, as discussed in Ref. [29]. This shows that our needed homogeneous field has the correct physical magnitude of the QED vacuum polarized by the hydrogen atom. We see that the *ad hoc* DFS demands a free field produced by the boundaries that is calculated to have a physically sensible order of magnitude. This approach to justify the DFS with a free field produced by the boundaries is similar to that of the stochastic electrodynamics of Refs. [30,31].

Finally, we mention a more radical alternative to justify our *ad hoc* DFS, by generalizing Dirac's theory such that the DFS would be derived *from principle*. This approach was taken by Eliezer and this generalization, henceforth called the Eliezer's setting (ES), is discussed in the excellent review of Ref. [17]. The ES involves the advanced interactions naturally, exactly in the same form of the DFS! Better still, the ES [17] contains an arbitrary parameter, and it would be highly desirable to experiment with stability analysis and the concept of resonant dissipation in the ES [17]. Even though the ES involves delay, advance, and third derivatives exactly like the DFS, the coefficients in the ES are never equal to those of the DFS. Our preliminary findings with the DFS suggest a future for this enterprise in the qualitative behavior of electromagneticlike dynamics, one that could describe QED by a stiff dynamical system with delay.

APPENDIX B: DARWIN AND THE SOFT COULOMBIAN MODES

In this appendix we calculate b_1 and b_2 of Eq. (2) as a function of m_1, m_2, and θ. The radial component of the electron's equation of motion along the circular orbit is [18]

$$\frac{m_1 b_1 r_b \theta^2}{\sqrt{1 - \theta^2 b_1^2}} = \frac{1}{S^3}\{[C^2 + \theta^2 S(S-1)][b_1 + b_2 \cos(\theta)] + \theta S[\sin(\theta) - \theta \cos(\theta)]b_2\}, \tag{B1}$$

where C and S are defined in Eqs. (13) and (14) respectively. Our Eq. (B1) is Eq. (3.2) of Ref. [18] after use of Eq. (4) and the identity

$$(1 - \theta^2 b_1^2)(1 - \theta^2 b_2^2) = C^2 + \theta^2(S - 2)S. \tag{B2}$$

The radial equation for the proton is obtained by exchanging the subindices 1 and 2 in Eq. (B1). There are three equations involving b_1, b_2, θ and r_b; (i) Eq. (B1); (ii) the equation for the proton, obtained by exchanging indices 1 and 2 in Eq. (B1), and (iii) the light-cone condition, Eq. (4)

$$\frac{m_1 b_1 r_b \theta^2}{\sqrt{1 - \theta^2 b_1^2}} = \frac{1}{S^3}\{[C^2 + \theta^2 S(S-1)][b_1 + b_2 \cos(\theta)] + \theta S[\sin(\theta) - \theta \cos(\theta)]b_2\},$$

$$\frac{m_2 b_2 r_b \theta^2}{\sqrt{1 - \theta^2 b_2^2}} = \frac{1}{S^3}\{[C^2 + \theta^2 S(S-1)][b_2 + b_1 \cos(\theta)] + \theta S[\sin(\theta) - \theta \cos(\theta)]b_1\}, \tag{B3}$$

$$b_1^2 + b_2^2 + 2b_1 b_2 \cos(\theta) = 1.$$

For small values of θ (atomic physics), we can solve Eqs. (B3) in a power series of θ with a symbolic manipulation software, yielding

$$b_1 = \frac{m_2}{M}\left(1 + \frac{\mu\theta^2}{2M}\right) + \theta^4 D(m_1, m_2) + \cdots,$$

$$b_2 = \frac{m_1}{M}\left(1 + \frac{\mu\theta^2}{2M}\right) + \theta^4 D(m_2, m_1) + \cdots, \tag{B4}$$

where

$$D(m_1, m_2) \equiv \left(\frac{\mu}{24M}\right)\left[\frac{12m_1^3 - 13m_2^3 - 5m_1 m_2^2 + 11m_2 m_1^2}{M^3}\right]. \tag{B5}$$

It is easy to continue this power series, but for the stiff limit in the atomic magnitude, even the θ^2 correction already gives a very small correction. The orbital frequency is determined by

$$\Omega = \frac{v_1}{b_1 r_b} = \mu\theta^3\left[1 + \left(\frac{1}{2} + \frac{\mu}{2M}\right)\theta^2 + \cdots\right]; \tag{B6}$$

the first term is Kepler's third law if we use $\theta = \Omega r_b$, and the next term is the Darwin correction. More information about the isolated two-body problem can be found in Refs. [32,22].

As an application of the above expansion, we calculate the soft Coulombian modes of Eq. (31) at a finite p by expanding up to $O(\theta^3)$

$$\det Z = \frac{M}{\mu}p^2(1 + p^2)\left[1 - \frac{\theta^2}{2}\left(1 - \frac{12\mu}{M}\right)\right] - \frac{2}{3}\frac{M}{\mu}p^3\left(p^2 + 2\frac{\mu}{M}\right). \tag{B7}$$

The soft roots of $\det Z = 0$ for Eq. (B7) are the Galilean translation mode $p = 0$ (a double root) and the oscillatory solutions $p \simeq \pm i$ that have a real part describing the radiative damping of the DFS, a familiar feature. We had partial success describing the atomic dynamics of helium with the Darwin approximation [33], and the tools of stability analysis used here were already used in Refs. [15,16]. The concept of resonant dissipation is new, and it is a generalization of the concept of a nonionizing dynamics of Ref. [33]. Unfortunately, the theory of Refs. [15,16,33] fails to describe discrete states for hydrogen because the soft Darwin modes are neutrally stable. As we have seen here, it is the stiff modes that equilibrate the dynamics, and those are beyond the Darwin approximation.

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